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Consistent Nonparametric Methods for Network Assisted Covariate Estimation

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Node Covariate Estimation

- Example: in a social network, each person has a vector of interests
  - desired products
  - preferred news topics
  - sporting interests
- Know a few people's interest vector from, say, their past tweets or comments. Unavailable or insufficient for others.
- **Problem: Can we infer their interests from a few people's known interests and the structure of the social network?**

Model

Latent Variable Models:

- Each node $i \in [n]$ has latent vector $\mathbf{z}_i \in \mathbb{R}^d$
- Network:

$$\mathbf{P}_{ij} := \mathrm{P}(\mathbf{A}_{ij} = 1 | \mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n) = \rho_n f(\mathbf{z}_i, \mathbf{z}_j; \boldsymbol{\Theta}) \quad \text{for all } i \neq j$$

  - $f(\cdot)$ is bounded in $[0,1]$ and has parameters $\boldsymbol{\Theta}$
  - $\rho_n = o(1)$ controls the sparsity of the graph
- Node Covariate:

$$\mathbf{X}_i = g(\mathbf{z}_i) + \boldsymbol{\epsilon}_i$$

  - $g : \mathbb{R}^d \to \mathbb{R}^p$ is bounded, and is Lipschitz
  - $\boldsymbol{\epsilon}_i$ are i.i.d. with uncorrelated elements, whose mean is 0 and variance is σ^2
- **Problem: given node covariates $\{\mathbf{X}_i \in \mathbb{R}^p; i \in S\}$ for a subset of nodes S, infer the node covariates of the remaining nodes $\{\mathbf{X}_i; i \in [n] \setminus S\}$.**

Related work

- Node similarity measures
  - Common neighbors and its weighted variants (Adamic/Adar), preferential attachment, resource allocation, Katz index, PageRank, SimRank, and graph neural networks, etc.
  - Only a few have consistency guarantees (**sarkar2011theoretical**; **sarkarchak2015**).
  - Our method constructs a similarity measure that provably works in sparser settings.
- Node classification
  - Methods based on random walks: label propagation, personalized PageRank, partially absorbing random walks, etc.
  - Node embeddings: represent nodes by vectors while retaining some network-based properties: DeepWalk, LINE, node2vec, spectral embedding etc.
    - Can train a classifier to predict the unseen labels with the embedding vectors.
    - Typically lack provable guarantees, but often work well in practice.
    - Spectral embedding is well studied and has provable guarantees, but limited to low-rank models.
    - Our method does not need low-rank assumption.

Model-Agnostic Algorithm

- Nonparametric estimator: $\hat{\mathbf{X}}_i = \dfrac{\sum_{j \in top_k(i)} \mathbf{W}_{ij} \cdot \mathbf{X}_j}{\sum_{j \in top_k(i)} \mathbf{W}_{ij}}$
  - $\mathbf{W}_{ij}$: measure of similarity between $\mathbf{z}_i$ and $\mathbf{z}_j$
    - Adjacency matrix $\mathbf{A}$: average of neighbors, but cannot distinguish in-cluster and out-of-cluster edges for stochastic blockmodel (SBM)
    - Common neighbor matrix $\mathbf{C}$: works only when average degree larger than $\sqrt{n}$
    - Distances between rows of $\mathbf{C}$: goes beyond common neighbors and can work for sparser graph – our proposed work
  - $top_k(i)$: set of nodes $j \in S$ with the largest $\mathbf{W}_{ij}$ values.
- Construct a new similarity measure

$$\mathbf{K}_{ij} = \sum_{k \neq i,j} \left[(\mathbf{C}_{ik}^2 - 2) 1(\mathbf{C}_{ik} \geq 2) + (\mathbf{C}_{jk}^2 - 2) 1(\mathbf{C}_{jk} \geq 2) - 2\mathbf{C}_{ik}\mathbf{C}_{jk} \right].$$

  - We prove that when average degreee grows faster than $n^{1/3}$, it concentrates to $\left(\sum_{k \neq i,j} \left((\mathbf{P}^2)_{ik} - (\mathbf{P}^2)_{jk}\right)^2\right)$, and can be used to pick nearest neighbors.
  - Proof sketch:
    - when $n\rho^2 \to 0$, $\mathrm{E}[\mathbf{C}_{ik}] \approx 0$ and $\mathbf{C}_{ik}$ does not concentrate
    - $\mathbf{C}_{ik}$ is well-approximated by a Poisson random variable with rate $\lambda_{ik} = (\mathbf{P}^2)_{ik} = O(n\rho^2)$.
    - indicator $1(\mathbf{C}_{ik} \geq 2)$ is true when $\mathbf{C}_{ik} = 2$ with probability $\approx \lambda_{ik}^2/2$, and $\mathbf{C}_{ik} > 2$ can be ignored
    - $\mathbf{C}_{ik}\mathbf{C}_{jk} = 1$ with probability $\approx \lambda_{ik}\lambda_{jk}$, with higher values having probabilities of a lower order.
    - we expect $\mathbf{K}_{ij} \approx \sum_k (2 \cdot (\lambda_{ik}^2/2 + \lambda_{jk}^2/2) - 2\lambda_{ik}\lambda_{jk}) = \sum_k (\lambda_{ik} - \lambda_{jk})^2$, which gives the desired concentration result.
- Algorithm summary:

CN-VEC

for $i \in [n] \setminus S$
- $dist(j) \leftarrow \mathbf{K}_{ij}$, for $j \in S$
- $top_k(i) \leftarrow k$ nodes with the smallest values of $dist(j)$
- $\hat{\mathbf{X}}_i \leftarrow \frac{1}{k} \sum_{j \in top_k(i)} \mathbf{X}_j$

  - We prove weak consistency result on CN-VEC when average degreee grows faster than $n^{1/3}$

Algorithm for low-rank models

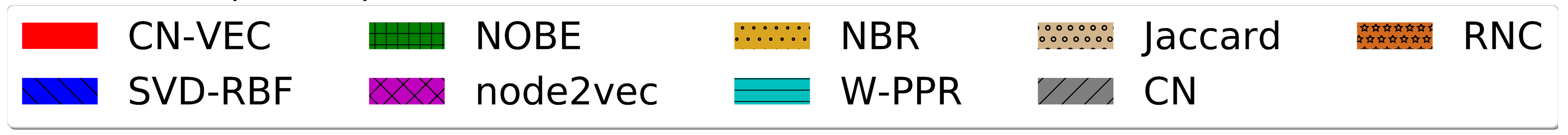
SVD-RBF

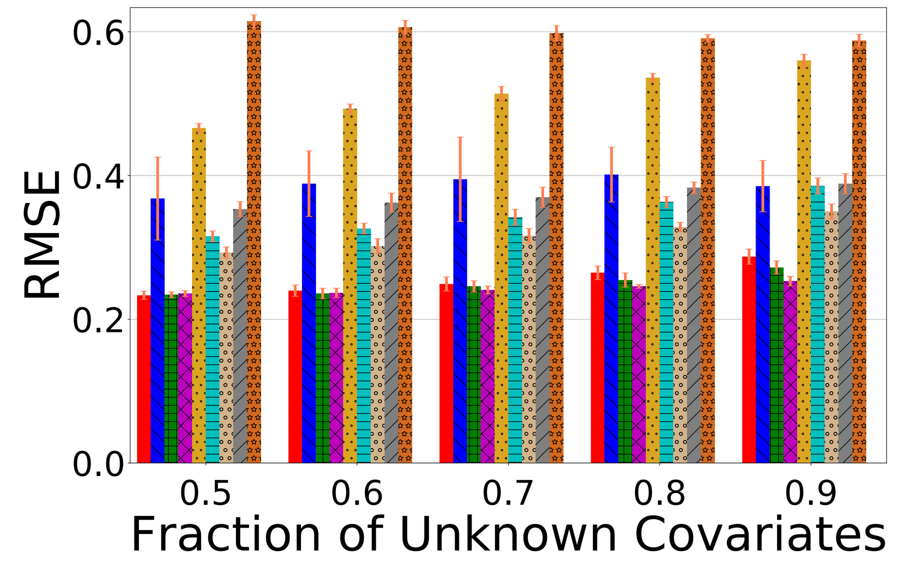
- $\hat{\mathbf{U}} \leftarrow$ top-d eigenvector matrix for $\mathbf{A}$
- $\hat{\mathbf{v}}_i \leftarrow i^{th}$ row of $\hat{\mathbf{U}}|\hat{\mathbf{E}}|^{1/2}$
- for $i \in [n] \setminus S$
  - $dist(j) \leftarrow \|\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_j\|$ for $j \in S$
  - $\hat{\mathbf{X}}_i \leftarrow \dfrac{\sum_{j \in S} K_\theta(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j)\, \mathbf{X}_j}{\sum_{j \in S} K_\theta(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j)}$, where $K_\theta(\mathbf{v}_1, \mathbf{v}_2) = \exp\left(-\dfrac{\|\mathbf{v}_1 - \mathbf{v}_2\|^2}{2\theta^2}\right)$

  - We prove uniform consistency result on SVD-RBF when average degreee grows faster than $\tilde{O}(\log n)$

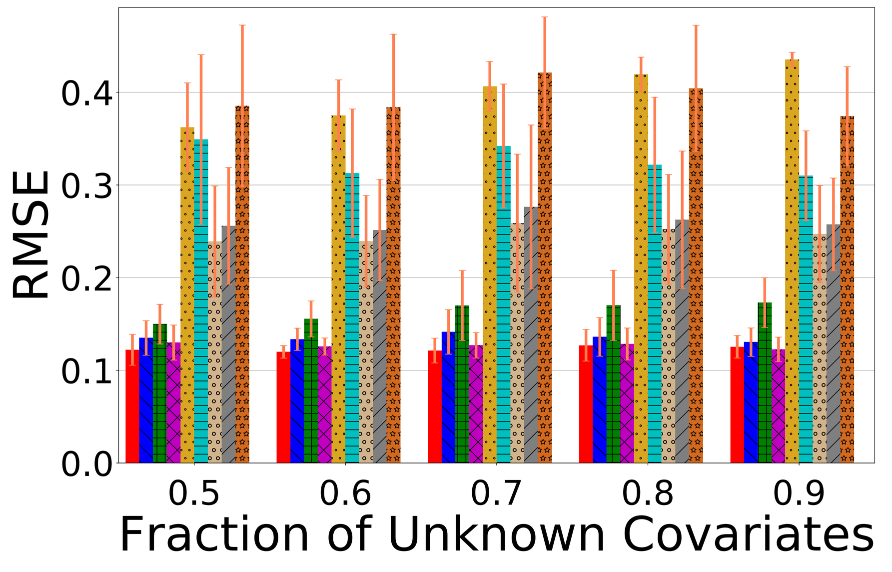
Simulation Experiments

- Comparing with embedding methods (NOBE, node2vec), neighborhood average (NBR), personalized pagerank (W-PPR), JACCARD, common neighbors (CN), regression with network cohesion (RNC)
- Simulate on latent space model (LSM), stochastic blockmodel (SBM), mixed-membership stochastic blockmodel (MMSB), rand random dot product model (RDPG)

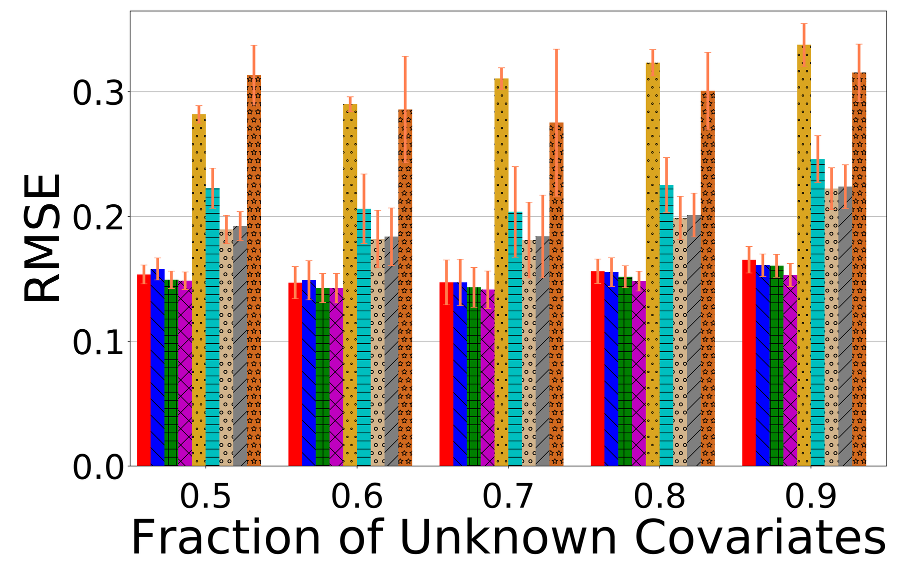




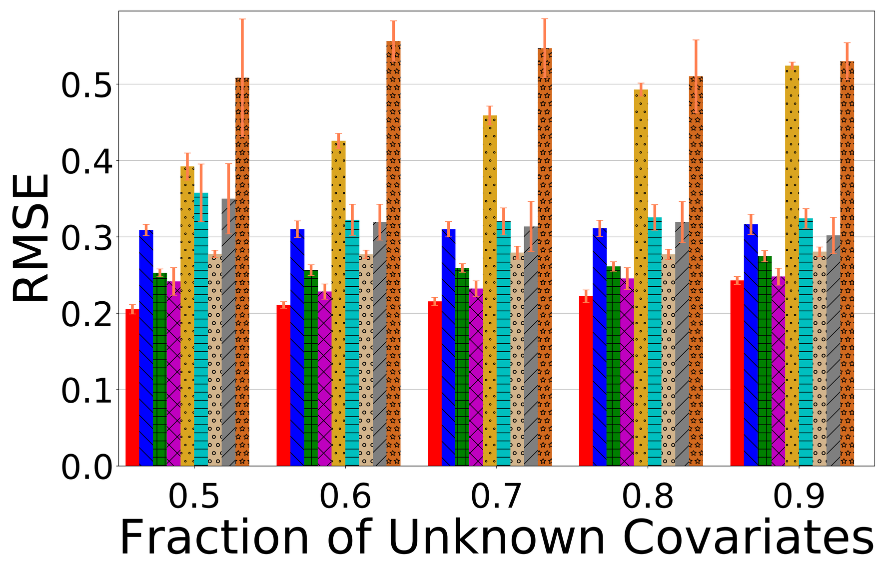
(a) LSM



(b) SBM



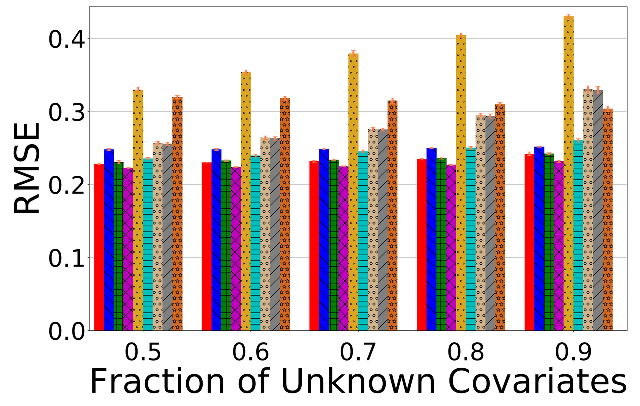
(c) MMSB

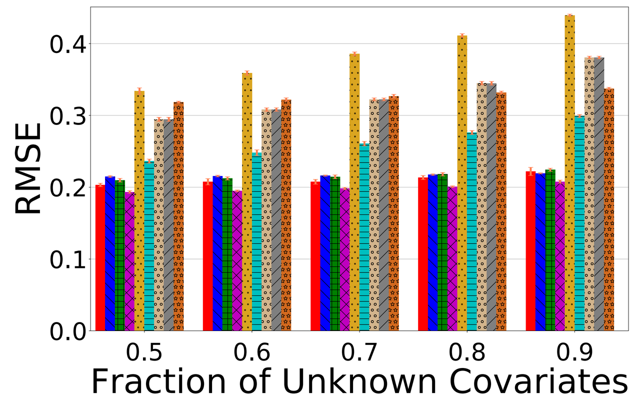


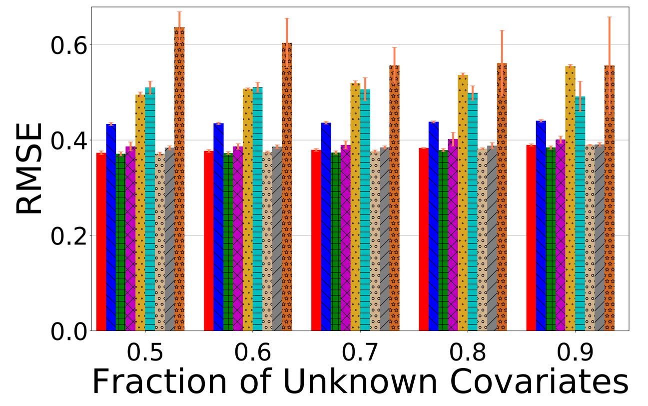
(d) RDPG

Real-world Network Results

- Citation networks (Cora and CiteSeer), and social network (Sinanet)
- Use topic distribution as node covariate







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